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A Discrete Multiple Scales Analysis of a Discrete Version of the Korteweg–de Vries Equation

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A more elaborate discrete multiple scales analysis than that used by Newell in 1977 is performed on the Zabusky–Kruskal discretization of the Korteweg–de Vries (KdV) equation. This eventually leads to a set of partial difference equations describing the modulational behavior of a small harmonic wave modulated by a slowly varying envelope. In the case of certain modes of the carrier wave, the multiple scales analysis breaks down, indicating that in these cases the numerical solution deviates in behavior from that of the KdV equation. Numerical experiments are reported which confirm this. © 1992 Academic Press, Inc.

1. INTRODUCTION

Recently Maritz and Schoombie [6] reported the occurrence of small amplitude, saw toothed wave packets when using the Zabusky–Kruskal finite difference scheme [15] to solve the Korteweg–de Vries (KdV) equation with rectangular pulse initial data. They also showed that a similar scheme for the modified Korteweg–de Vries (MKdV) equation admits saw toothed solitary wave packets with near soliton behavior. Sloan [10] also considered the Zabusky–Kruskal scheme for the KdV, showing that the presence of the dispersive term causes modulational instabilities.

In this paper we continue this study of the Zabusky–Kruskal scheme for the KdV equation. We now focus our attention on the modulation properties of this finite difference scheme when the initial datum consists of a small amplitude harmonic wave with long, slow modulations.

Our method of analysis is a discrete version of the well-known multiple scales technique [7, 14]. This technique has often been used to establish a modulation theory for certain nonlinear dispersive wave equations in the case of slow, weak modulations [3–5]. In the case of the KdV equation, such an analysis shows that the envelopes of the modulated waves are governed by the nonlinear Schrödinger equation [1].

As far as we know, the only multiple scales analysis for partial difference equations reported in the literature is that of Newell [8]. Newell's analysis has been successfully applied by both himself and others (like Stuart [11]) to various nonlinear partial difference equations. It does not, however, give a clear picture of the relationships between partial differences with respect to the various time and space scales. We therefore devised our own method of analysis, which is quite analogous to the analysis for differential equations. For this purpose we derived a special difference identity which serves the same purpose as the chain rule for derivatives.

The end result of our multiple scales analysis is a system of difference equations for the wave envelope. One of these is consistent with the nonlinear Schrödinger equation for a low wave number carrier wave.

Obviously these modulation equations can be valuable in studies of modulational instabilities and spurious solutions of the difference scheme. In this paper we report only one type of spurious solution which occurs for certain choices of the parameters in the KdV equation and for specific wave numbers of the carrier wave.

In Section 2 a brief review is given of the modulational theory of the continuous equation, as derived in [1, 16, 12]. We point out that the method used by Zakharov and Kuznetsov [16] and Tracy *et al.* [12] is particularly suited for adaptation to the discrete case. In Section 3 we obtain the discrete analogs of these equations for the Zabusky–Kruskal scheme. Section 4 contains a discussion of the modulations of a saw toothed wave, and in Section 5 a few numerical results are reported. In Section 6, we give our conclusions and point out a few other possibilities.

2. ANALYSIS OF THE KDV EQUATION

For purposes of comparison we shall give a review of known results concerning long slow modulations of small monochromatic harmonic waves of which the evolution is governed by the KdV equation. Full details may be found in [1, 16, 12].

Consider the KdV equation in the form

$$u_t + \eta u_x + \zeta u u_x + \gamma u_{xxx} = 0, \tag{1}$$

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where the subscripts denote partial differentiation as usual and where η, ζ, and γ are constants, with $\gamma \neq 0$. It is furthermore assumed that suitable initial data

$$u(x, 0) = \varepsilon f(x), \qquad f(x) = O(1), \tag{2}$$

are prescribed, as well as the periodic conditions

$$\begin{aligned} u(x \pm L, t) &= u(x, t), \\ f(x \pm L) &= f(x), \qquad t > 0, \qquad x \in \mathbb{R}, \end{aligned} \tag{3}$$

where ε is a small, real, positive parameter.

We shall review two versions of a multiple scales analysis, which are completely equivalent. The first is described in [1] and involves the generation of a hierarchy of linear partial difference equations from which secular terms are removed to ensure a bounded solution of (1).

For this purpose it is assumed that the solution $u(x, t)$ of (1) can be expanded in the form

$$u(x, t) = \sum_{n=1}^{\infty} \varepsilon^n u^{(n)}(X_0, X_1, T_0, T_1, T_2), \tag{4}$$

where

$$X_k = \varepsilon^k x, \qquad k = 0, 1, \tag{5}$$

and

$$T_k = \varepsilon^k t, \qquad k = 0, 1, 2. \tag{6}$$

For $k = 0$ we have the fast scales in time and space, and for the higher values of k we have progressively longer space and slower time scales.

We also use the expansions

$$\frac{\partial}{\partial x} = \sum_{n=0}^{1} \varepsilon^n \frac{\partial}{\partial X_n} \tag{7}$$

and

$$\frac{\partial}{\partial t} = \sum_{n=0}^{2} \varepsilon^n \frac{\partial}{\partial T_n} \tag{8}$$

based on the chain rule for partial differentiation. When substituting (4) through (8) into (1) and collecting terms containing equal powers of ε, we generate a hierarchy of equations of which the first three members are

$$Lu^{(1)} = 0 \tag{9}$$

$$Lu^{(2)} = -u^{(1)}_{T_1} - \eta u^{(1)}_{X_1} - \zeta u^{(1)} u^{(1)}_{X_0} - 3\gamma u^{(1)}_{X_0 X_0 X_1} \tag{10}$$

$$\begin{aligned} Lu^{(3)} = &-u^{(2)}_{T_1} - u^{(1)}_{T_2} \\ &- \zeta[(u^{(1)}u^{(2)})_{X_0} + \tfrac{1}{2} u^{(1)}_{X_1}] \\ &- 3\gamma(u^{(2)}_{X_0 X_0 X_1} + u^{(1)}_{X_0 X_1 X_1}), \end{aligned} \tag{11}$$

where L is the linear operator,

$$L := \frac{\partial}{\partial T_0} + \eta \frac{\partial}{\partial X_0} + \gamma \frac{\partial^3}{\partial X_0^3}. \tag{12}$$

We now consider a solution of (9) of the form

$$u^{(1)} = A(X_1, T_1, T_2) \exp(i\theta) + \text{complex conjugate}, \tag{13}$$

where

$$\theta = kX_0 - \omega T_0, \tag{14}$$

and where the carrier wave number k is related to the carrier wave frequency ω by the *linear dispersion relation*

$$\omega = \eta k - \gamma k^3. \tag{15}$$

We shall restrict k to nonzero values only, since the trivial case $k = 0$ requires special treatment in the analysis which follows and is of little interest to us in any case.

Recall that $u(x, t)$ must satisfy the periodic conditions specified in (3), and to ensure this with respect to at least the fast scale in space we must restrict the wave number k to the values

$$k = k_m = 2\pi m/L, \qquad m = 1, 2, \ldots. \tag{16}$$

We now substitute (13) into (10) and find that in order to obtain a bounded solution $u^{(2)}$, we have to remove secular terms by imposing the condition

$$A_{T_1} + C_g A_{X_1} = 0, \tag{17}$$

where

$$C_g := \frac{d\omega}{dk} = \eta - 3\gamma k^2 \tag{18}$$

is the *linear group velocity* associated with the operator L. We then find that (10) has a solution of the form

$$\begin{aligned} u^{(2)} = &[\zeta/(6\gamma k^2)][A^2 \exp(2i\theta) + A^{*2} \exp(-2i\theta)] \\ &+ B(X_1, T_1, T_2), \end{aligned} \tag{19}$$

where B is a function yet to be determined.

Substituting (13) and (19) into (11), we find that secular terms can be removed by imposing the conditions

$$B_{T_1} + \eta B_{X_1} + \zeta(|A|^2)_{X_1} = 0, \tag{20}$$

$$A_{T_2} + i[\zeta^2/(6\gamma k)]\, A\, |A|^2 + i\zeta k B A + 3i\gamma A_{X_1 X_1} = 0, \tag{21}$$

as well as the condition that B should be a real function.

Equations (17), (20), and (21) are now the required modulation equations, describing the behavior of the envelope A. Since the function B is not uniquely determined by the equation above, we are free to impose additional conditions. Such conditions can be chosen for convenience, since the only effect they could have is to cause unbounded growth on a time scale of $O(1/\varepsilon^3)$ in $u^{(4)}$ and further terms of (4). For small enough ε, this can be neglected for our purposes.

Thus, from (17),

$$A = A(\bar{X}, T_2), \tag{22}$$

where

$$\bar{X} = X_1 - C_g T_1. \tag{23}$$

If we make the physically reasonable assumption that B also satisfies (17), (20) is seen to have a solution

$$B(\bar{X}, T_2) = -(\zeta/3\gamma k^2)\,|A|^2 \tag{24}$$

so that (21) can be rewritten as

$$A_{T_2} + 3i\gamma k A_{\bar{X}\bar{X}} - [i\zeta^2/(6\gamma k)]\,A\,|A|^2 = 0. \tag{25}$$

This is the nonlinear Schrödinger equation in the variables $\bar{X}$ and T_2. Note that (17) describes a *linear* modulation property of (1), whereas (25) gives some information about the modulation effects of the *nonlinear* terms in (1). Thus (17) implies a modulational envelope moving at group velocity C_g on the time scale T_1, whereas further nonlinear modulations on the slower time scale T_2 are governed by the nonlinear Schrödinger equation (25). It can in fact be shown, by using the known properties of the nonlinear Schrödinger equation, that the envelope A is stable in all its modes. (See, for instance, [13]).

Equations (13) through (25) can also be derived in a different way, using the version of the multiple scales analysis used in [16, 12]. Here we start with the expansion

$$u(x, t) = \sum_{r=-\infty}^{\infty} u_r(X_1, T_1, T_2, \varepsilon)\, e^{ir\theta}, \tag{26}$$

where

$$u_r = \varepsilon^{\delta_r} v_r(X_1, T_1, T_2, \varepsilon) \tag{27}$$

and

$$\delta_0 = 2, \qquad \delta_r = \delta_{-r} = |r|, \qquad r \neq 0, \tag{28}$$

and

$$v_0 = V_0(X_1, T_1, T_2), \tag{29}$$

$$v_1 = V_1(X_1, T_1, T_2) \tag{30}$$

and, for $r \geqslant 2$,

$$\begin{aligned} v_r &= v^*_{-r} \\ &= V_r(X_1, T_1, T_2) + \sum_{s=r}^{\infty} \varepsilon^{s+1-r} W_{rs}(X_1, T_1, T_2). \end{aligned} \tag{31}$$

Note that θ is still given by (14) and k by (16).

We now substitute the expansion (26) into (1). When doing so, we now use the expansions

$$\frac{\partial}{\partial t} = -ir\omega + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} \tag{32}$$

and

$$\frac{\partial}{\partial x} = irk + \varepsilon \frac{\partial}{\partial X_1}, \tag{33}$$

instead of the more general (7) and (8). Putting the coefficient of each $e^{ir\theta}$ equal to zero, we find that (1) is equivalent to the following system of equations:

$$\begin{aligned} &-ir\omega u_r + \varepsilon(u_r)_{T_1} + \varepsilon^2(u_r)_{T_2} + ir\eta k u_r + \eta\varepsilon(u_r)_{X_1} \\ &\quad + \gamma\left[irk + \varepsilon\frac{\partial}{\partial X_1}\right]^3 u_r \\ &\quad + \zeta \sum_{s=-\infty}^{\infty} [iksu_s + \varepsilon(u_s)_{X_1}]\, u_{r-s} = 0. \end{aligned} \tag{34}$$

We now wish to have (1), and therefore (34) for each r, satisfied up to terms of $O(\varepsilon^3)$. We note that it is therefore only necessary to consider (34) for $|r| \leqslant 3$.

First put $r = 0$ in (34). The term of lowest order in ε is then $O(\varepsilon^3)$, and putting the coefficient of ε^3 equal to zero yields the equation

$$(V_0)_{T_1} + \eta(V_0)_{X_1} + \zeta(|V_1|^2)_{X_1} = 0. \tag{35}$$

Next put $r = 1$ in (34). The requirement that the coefficient of ε should vanish yields the linear dispersion equation (15). By next putting the coefficients of ε^2 and ε^3 to zero as well, the following two equations are obtained:

$$(V_1)_{T_1} + C_g(V_1)_{X_1} = 0, \tag{36}$$

$$(V_1)_{T_2} + 3i\gamma k(V_1)_{X_1X_1} + ik\zeta[V_1V_0 + V_1^*V_2] = 0. \tag{37}$$

Equation (34) with $r = 2$ contains $O(\varepsilon^2)$ and $O(\varepsilon^3)$ terms. From the former we obtain, with the aid of the linear dispersion equation (15),

$$V_2 = \frac{\zeta V_1^2}{6\gamma k^2}, \tag{38}$$

and from the latter it follows, after some manipulation and using (15), (36), and (38):

$$W_{22} = \frac{i\zeta[V_1^2]_{X_1}}{6\gamma k^3}. \tag{39}$$

Finally the case $r = 3$ in (34) involves only $O(\varepsilon^3)$ terms, and from these, with the aid of (15) and (38), follows the equation

$$V_3 = \frac{\zeta^2 V_1^3}{48\gamma^2 k^4}. \tag{40}$$

If we now identify V_1 with A, and V_0 with B in the previous version of the multiple scales analysis, we see that (35), (36) is the same as (20), (17), respectively, and when we combine (37) and (38) we obtain (21). Thus exactly the same results are obtained as before.

3. MULTIPLE SCALES ANALYSIS OF THE ZABUSKY–KRUSKAL SCHEME

Before we analyze the Zabusky–Kruskal discretization of (1) in a manner analogous to the method in [16, 12] described in the previous section, it is first necessary to introduce some notation.

In order to discretize in space, we divide the interval $[0, L]$ into N subintervals, using the grid length

$$h = L/N. \tag{41}$$

We shall assume N to be even throughout this paper for the sake of definiteness; odd N will require a few straightforward modifications in the analysis which follows.

Also introducing a time step τ, we use the symbol u_j^n to denote the solution of the difference scheme at $x = hj$ and $t = n\tau$, where j and n are integers, i.e.,

$$u_j^n = u(hj, n\tau). \tag{42}$$

We next introduce the shift operators E_x and E_t in space and time, respectively, defined by

$$E_x f(x, t) = f(x + h, t) \tag{43}$$

$$E_t f(x, t) = f(x, t + \tau), \tag{44}$$

so that

$$E_x u_j^n = u_{j+1}^n; \qquad E_t u_j^n = u_j^{n+1}. \tag{45}$$

We next define the following *divided difference* operators:

$$\Delta_x := (E_x - 1)/h \tag{46}$$

$$\nabla_x := (1 - E_x^{-1})/h \tag{47}$$

$$\delta_x := (\Delta_x + \nabla_x)/2 \tag{48}$$

$$\mu_x := (E_x + 1 + E_x^{-1})/3 = 1 + h(\Delta_x - \nabla_x)/3 \tag{49}$$

$$\Delta_t := (E_t - 1)/\tau \tag{50}$$

$$\nabla_t := (1 - E_t^{-1})/\tau \tag{51}$$

$$\delta_t := (\Delta_t + \nabla_t)/2. \tag{52}$$

The Zabusky–Kruskal scheme can then be written in the form [15]:

$$\delta_t u_j^n + [\eta + \zeta(\mu_x u_j^n)]\, \delta_x u_j^n + \gamma \delta_x \Delta_x \nabla_x u_j^n = 0. \tag{53}$$

In addition we impose the periodic conditions

$$u_{j\pm N}^n = u_j^n \tag{54}$$

and prescribe initial data of the form

$$u_j^0 = \varepsilon f_j, \qquad f_j = O(1), \qquad f_{j\pm N} = f_j, \tag{55}$$

where ε is a small, real, positive parameter as before.

We next define the following multiple-scale coordinates in space and time, analogously to (5) and (6),

$$X_p = \varepsilon^p hj, \qquad p = 0, 1, \tag{56}$$

$$T_p = \varepsilon^p n\tau, \qquad p = 0, 1, 2. \tag{57}$$

Obviously we will require equations analogous to (7) and (8). We will now proceed to establish these.

First we have to define partial difference operators corresponding to the various variables X_p and T_p. We define the partial shift operators E_{T_p} and E_{X_p}:

$$E_{X_p} f(..., X_p, ...) = f(..., X_p + \varepsilon^p h, ...) \tag{58}$$

$$E_{T_p} f(..., T_p, ...) = f(..., T_p + \varepsilon^p \tau, ...). \tag{59}$$

Thus if we have a discrete function $u(X_0, X_1, T_0, T_1, T_2)$, then

$$E_t u = E_{T_0} E_{T_1} E_{T_2} u \tag{60}$$

and

$$E_x u = E_{X_0} E_{X_1} u. \tag{61}$$

We also define

$$\Delta_{T_p} := (E_{T_p} - 1)/(\varepsilon^p \tau) \tag{62}$$

$$\nabla_{T_p} := (1 - E_{T_p}^{-1})/(\varepsilon^p \tau) \tag{63}$$

$$\Delta_{X_p} := (E_{X_p} - 1)/(\varepsilon^p h) \tag{64}$$

$$\nabla_{X_p} := (1 - E_{X_p}^{-1})/(\varepsilon^p h) \tag{65}$$

$$\delta_{X_p} := (\Delta_{X_p} + \nabla_{X_p})/2 \tag{66}$$

$$\delta_{T_p} := (\Delta_{T_p} + \nabla_{T_p})/2. \tag{67}$$

We now state the following lemma, which is the keystone to our multiple scales analysis:

LEMMA 3.1. *Let for any function*

$$f = f(X_0, X_1, ..., X_n, T_0, T_1, ..., T_m), \tag{68}$$

where

$$X_p = \varepsilon^p hj, \qquad p = 0, 1, ..., n, \tag{69}$$

$$T_p = \varepsilon^p n\tau, \qquad p = 0, 1, ..., m, \tag{70}$$

and let E_{X_p} and E_{T_p} be defined as in (58) *and* (59). *Then the divided difference operators defined by* (62) *through* (65) *satisfy the relations*:

$$\Delta_x = \Delta_{X_0} + \sum_{p=1}^{n} \varepsilon^p \Delta_{X_p} E_{X_0} E_{X_1} \cdots E_{X_{p-1}} \tag{71}$$

$$\nabla_x = \nabla_{X_0} + \sum_{p=1}^{n} \varepsilon^p \nabla_{X_p} E_{X_0}^{-1} E_{X_1}^{-1} \cdots E_{X_{p-1}}^{-1} \tag{72}$$

$$\Delta_t = \Delta_{T_0} + \sum_{p=1}^{m} \varepsilon^p \Delta_{T_p} E_{T_0} E_{T_1} \cdots E_{T_{p-1}} \tag{73}$$

$$\nabla_t = \nabla_{T_0} + \sum_{p=1}^{m} \varepsilon^p \nabla_{T_p} E_{T_0}^{-1} E_{T_1}^{-1} \cdots E_{T_{p-1}}^{-1}. \tag{74}$$

Proof. We shall only prove (71). The rest of the relations are proved similarly. First note that

$$E_x = E_{X_0} E_{X_1} \cdots E_{X_n}. \tag{75}$$

Then

$$\begin{aligned} \Delta_{X_0} &+ \sum_{p=1}^{n} \varepsilon^p \Delta_{X_p} E_{X_0} E_{X_1} \cdots E_{X_{p-1}} \\ &= \frac{1}{h}\left[\sum_{p=1}^{n} (E_{X_0} E_{X_1} \cdots E_{X_p} - E_{X_0} E_{X_1} \cdots E_{X_{p-1}}) + E_{X_0} - 1\right] \\ &= \frac{1}{h}[E_{X_0} E_{X_1} \cdots E_{X_n} - 1] \\ &= \frac{1}{h}(E_x - 1) = \Delta_x. \quad \blacksquare \end{aligned} \tag{76}$$

Lemma 3.1 now gives us the expansions to use instead of the chain rules (7) and (8), using $m = 2$ and $n = 1$. The expansions we shall need in our analysis, can then be written in the form

$$\Delta_x = \Delta_{X_0} + \varepsilon \bar{\Delta}_{X_1} \tag{77}$$

$$\nabla_x = \nabla_{X_0} + \varepsilon \bar{\nabla}_{X_1} \tag{78}$$

$$\Delta_t = \Delta_{T_0} + \varepsilon \bar{\Delta}_{T_1} + \varepsilon^2 \bar{\Delta}_{T_2} \tag{79}$$

$$\nabla_t = \nabla_{T_0} + \varepsilon \bar{\nabla}_{T_1} + \varepsilon^2 \bar{\nabla}_{T_2} \tag{80}$$

$$\delta_x = \delta_{X_0} + \varepsilon \bar{\delta}_{X_1} \tag{81}$$

$$\delta_t = \delta_{T_0} + \varepsilon \bar{\delta}_{T_1} + \varepsilon^2 \bar{\delta}_{T_2} \tag{82}$$

$$\mu_x = \mu_{X_0} + \varepsilon \bar{\mu}_{X_1}, \tag{83}$$

where (77) through (83) were obtained from (48), (49), (52), and (71) through (74), and where expressions for the barred operators may be obtained directly from (71) through (74).

We are now ready to commence with the discrete multiple scales analysis of (53). We start with an expansion similar to (26), but in this case we have to take into account the effect of aliasing; i.e., only a finite number of modes can be resolved on a grid of discrete points in [0, L].

First let

$$\theta = khj - \Omega n\tau, \tag{84}$$

where k is now limited to the *finite* set of values

$$k = \frac{2\pi m}{L}, \qquad m = -N/2 + 1, ..., N/2, \tag{85}$$

due to aliasing.

Next let

$$u_j^n = \sum_{r=-[l/2]}^{[l/2]} c_r u_r(X_1, T_1, T_2, \varepsilon)\, e^{ir\theta}, \tag{86}$$

where l is obtained by letting s and l be the integers with least absolute values such that

$$\frac{m}{N} = \frac{s}{l} \tag{87}$$

and

$$[l/2] = \begin{cases} l/2 & \text{if } l \text{ even} \\ (l-1)/2 & \text{if } l \text{ odd,} \end{cases} \tag{88}$$

and where

$$c_r = \begin{cases} 1 & \text{if } |r| < l/2 \\ 1/2 & \text{if } |r| = l/2. \end{cases} \tag{89}$$

It is easy to see that the exponentials

$$\exp(irkhj) = \exp\left[ir\left(\frac{2\pi m}{N}\right)j\right] = \exp\left[ir\left(\frac{2\pi s}{l}\right)j\right] \tag{90}$$

can only take on l different values, corresponding to

$$r = \begin{cases} -l/2+1, ..., l/2 & \text{if } l \text{ even} \\ -(l-1)/2, ..., (l-1)/2 & \text{if } l \text{ odd,} \end{cases} \tag{91}$$

hence the summation range used in (86). In the case where l is even, the exponential corresponding to $r = l/2$ is exactly the same as the one corresponding to $r = -l/2$, and therefore the coefficients c_r are usually introduced (see [2]) to write (86) in a more symmetric form.

We shall also have occasion to use the fact that products of the form

$$P = \left(\sum_{r=-[l/2]}^{[l/2]} c_r U_r \exp(irkhj)\right) \times \left(\sum_{r=-[l/2]}^{[l/2]} c_r V_r \exp(irkhj)\right) \tag{92}$$

can be written in the convolution form

$$P = \sum_{n=-[l/2]}^{[l/2]} \sum_{m=-[l/2]}^{[l/2]} c_n c_m U_n V_{m-n} \exp imkhj. \tag{93}$$

As in the continuous case, we put

$$u_r = \varepsilon^{\delta_r} v_r(X_1, T_1, T_2, \varepsilon), \tag{94}$$

where

$$\delta_0 = 2, \qquad \delta_r = |r|, \qquad |r| = 1, 2, ..., [l/2], \tag{95}$$

and

$$v_0 = V_0(X_1, T_1, T_2), \tag{96}$$

$$v_1 = V_1(X_1, T_1, T_2) = v^*_{-1}, \tag{97}$$

$$v_r = v^*_{-r} = V_r(X_1, T_1, T_2) + \sum_{p=r}^{\infty} \varepsilon^{p+1-r} W_{rp}(X_1, T_1, T_2), \quad r = 2, ..., [l/2]. \tag{98}$$

We now wish to substitute the expansion (86) into the finite difference scheme (53). In this process we make use of (77) through (83), as well as the expansions

$$E_{T_1} = 1 + \varepsilon\tau\delta_{T_1} + \varepsilon^2\tau^2\Delta_{T_1}\nabla_{T_1}/2, \tag{99}$$

$$E_{T_1}^{-1} = 1 - \varepsilon\tau\delta_{T_1} + \varepsilon^2\tau^2\Delta_{T_1}\nabla_{T_1}/2, \tag{100}$$

from which follows

$$(E_{T_1}\Delta_{T_2} + E_{T_1}^{-1}\nabla_{T_2})/2 = \delta_{T_2} + \varepsilon^2\tau^2\Delta_{T_1}\nabla_{T_1}\delta_{T_2}/2 + \varepsilon^3\tau^2\delta_{T_1}\Delta_{T_2}\nabla_{T_2}/2 \tag{101}$$

$$(E_{T_1}\Delta_{T_2} - E_{T_1}^{-1}\nabla_{T_2})/2 = \varepsilon\tau\delta_{T_1}\delta_{T_2} + \varepsilon^4\tau^3\Delta_{T_1}\nabla_{T_1}\Delta_{T_2}\nabla_{T_2}/4 + \varepsilon^2\tau\Delta_{T_2}\nabla_{T_2}/2 \tag{102}$$

as well as

$$\Delta_{X_1} = \delta_{X_1} + \varepsilon h\Delta_{X_1}\nabla_{X_1} \tag{103}$$

$$\nabla_{X_1} = \delta_{X_1} - \varepsilon h\Delta_{X_1}\nabla_{X_1}, \tag{104}$$

from which follows

$$\delta_{X_1}^2 = \Delta_{X_1}\nabla_{X_1} + \varepsilon^2h^2\Delta_{X_1}^2\nabla_{X_1}^2/4. \tag{105}$$

We find that, up to terms of $O(\varepsilon^3)$,

$$\delta_t(u_r e^{ir\theta}) = P_r(u_r e^{ir\theta}) \tag{106}$$

$$\delta_x(u_r e^{ir\theta}) = Q_r(u_r e^{ir\theta}) \tag{107}$$

$$\mu_x(u_r e^{ir\theta}) = S_r(u_r e^{ir\theta}) \tag{108}$$

$$\delta_x\Delta_x\nabla_x(u_r e^{ir\theta}) = T_r(u_r e^{ir\theta}), \tag{109}$$

where, up to terms of $O(\varepsilon^2)$,

$$P_r = -is_\tau/\tau + \varepsilon c_\tau\delta_{T_1} + \varepsilon^2(c_\tau\delta_{T_2} - i\tau s_\tau\Delta_{T_1}\nabla_{T_1}/2) \tag{110}$$

$$Q_r = is_h/h + \varepsilon c_h\delta_{X_1} + i\varepsilon^2hs_h\Delta_{X_1}\nabla_{X_1}/2 \tag{111}$$

$$S_r = (1+2c_h)/3 + (2i\varepsilon h/3)\, s_h\delta_{X_1} + (\varepsilon^2h^2/3)\, c_h\Delta_{X_1}\nabla_{X_1} \tag{112}$$

$$T_r = (2i/h^3)\, s_h(c_h-1) + (2\varepsilon/h^2)(c_h-1)(2c_h+1)\,\delta_{X_1} + (i\varepsilon^2/h)\, s_h(4c_h-1)\,\Delta_{X_1}\nabla_{X_1}, \tag{113}$$

where

$$s_h = \sin(rkh), \qquad c_h = \cos(rkh) \tag{114}$$

and

$$s_\tau = \sin(r\Omega\tau), \qquad c_\tau = \cos(r\Omega\tau). \tag{115}$$

After substituting (86) into (53) and putting coefficients of $c_r \exp(ir\theta)$ equal to zero, we obtain the system of equations

$$\varepsilon^{\delta_r} P_r v_r + \eta \varepsilon^{\delta_r} Q_r v_r + \zeta \sum_{w=-[l/2]}^{[l/2]} c_w (S_w v_w) \times (Q_{r-w} v_{r-w}) \, \varepsilon^{\delta_w + \delta_{r-w}} + \gamma \varepsilon^{\delta_r} T_r v_r = 0, \qquad r = 0, 1, ..., [l/2]. \tag{116}$$

As in the continuous case, we now wish to satisfy (53), and therefore the system (116), up to terms in ε^3. Thus we need only consider the cases $r = 0$, 1, and 2 in (116).

When we put $r = 0$, we find that the lowest order term in ε is $O(\varepsilon^3)$, and putting this equal to zero yields the equation

$$\delta_{T_1} V_0 + \eta \delta_{X_1} V_0 + (\zeta c_1/3)(\cos(kh) + 2) \times (V_1 \delta_{X_1} V_1^* + V_1^* \delta_{X_1} V_1) = 0. \tag{117}$$

We can simplify this equation somewhat by making use of the discrete analog of the product rule of differentiation, which we state in the form of the following lemma:

LEMMA 3.2. *If $V = V(X_1)$ and $W = W(X_1)$, then*

$$\delta_{X_1}(VW) = (E_{X_1} V) \, \delta_{X_1} W + (E_{X_1}^{-1} W) \, \delta_{X_1} V. \tag{118}$$

Furthermore,

$$\delta_{X_1}(VW) = V \delta_{X_1} W + W \delta_{X_1} V + \varepsilon h (\Delta_{X_1} V \delta_{X_1} W - \nabla_{X_1} W \delta_{X_1} V). \tag{119}$$

Proof. The proof of (118) is straightforward:

$$\begin{aligned} 2\varepsilon h \delta_{X_1}(VW) &= E_{X_1} V E_{X_1} W - E_{X_1}^{-1} V E_{X_1}^{-1} W \\ &= E_{X_1} V (E_{X_1} W - E_{X_1}^{-1} W) \\ &\quad + E_{X_1}^{-1} W (E_{X_1} V - E_{X_1}^{-1} V) \\ &= 2\varepsilon h [(E_{X_1} V) \, \delta_{X_1} W + (E_{X_1}^{-1} W) \, \delta_{X_1} V]. \end{aligned} \tag{120}$$

We then write

$$\begin{aligned} &\delta_{X_1}(VW) - V \delta_{X_1} W - W \delta_{X_1} V \\ &\quad = [(E_{X_1} - 1) V] \, \delta_{X_1} W + [(E_{X_1}^{-1} - 1] W \delta_{X_1} V \\ &\quad = \varepsilon h [\Delta_{X_1} V \delta_{X_1} W - \nabla_{X_1} W \delta_{X_1} V], \end{aligned} \tag{121}$$

from which (119) follows immediately. ∎

Returning to (117), we see that, according to (119),

$$V_1 \delta_{X_1} V_1^* + V_1^* \delta_{X_1} V_1 = \delta_{X_1}(|V_1|^2) + O(\varepsilon), \tag{122}$$

and since (117) was obtained from the coefficient of an $O(\varepsilon^3)$ term in (116), the $O(\varepsilon)$ term in (122) may be neglected, so that (117) becomes

$$\delta_{T_1} V_0 + \eta \delta_{X_1} V_0 + (\zeta c_1/3)(\cos(kh) + 2) \times \delta_{X_1}(|V_1|^2) = 0. \tag{123}$$

We next put $r = 1$ in (116), and putting the coefficient of ε equal to zero, we obtain the discrete linear dispersion equation

$$(\sin \Omega\tau)/\tau = \eta (\sin kh)/h + (2\gamma/h^3) \times \sin kh (\cos kh - 1). \tag{124}$$

We may note at this point that if we perform a standard Von Neuman analysis on the linearized Zabusky–Kruskal scheme

$$\delta_t u_j^n + \eta \delta_x u_j^n + \gamma \delta_x \Delta_x \nabla_x u_j^n = 0; \tag{125}$$

i.e., if we put

$$u_j^n = r^n \exp(ikhj), \tag{126}$$

then we find that the amplification factor r is given by

$$r = -i\tau a(kh) \pm \sqrt{1 - \tau^2 a(kh)^2}, \tag{127}$$

where $a(kh)$ is the right-hand side of (124). Thus the requirement $|r| \leqslant 1$ for linear stability leads to

$$|\tau a(kh)| \leqslant 1, \tag{128}$$

and we see that in terms of (124) this simply means that we require that

$$|\sin \Omega\tau| \leqslant 1 \tag{129}$$

or, in other words, that Ω should be real. It is well known [8, 9] that the condition (128) leads to the following necessary and sufficient conditions for linear stability of (53):

$$\tau \leqslant 2h^3/(3\sqrt{3}\,\gamma) \tag{130}$$

and

$$\tau \leqslant h/\eta. \tag{131}$$

Returning to our multiple scales analysis, we next put the coefficient of ε^2 equal to zero to obtain the equation

$$\delta_{T_1} V_1 + V_g \delta_{X_1} V_1 = 0, \tag{132}$$

where

$$V_g := \frac{d\Omega}{dk} = \eta \cos kh + (2\gamma/h^2)[\cos(2kh) - \cos kh] \qquad (133)$$

is the discrete linear group velocity. Finally, when we put the coefficient of ε^3 equal to zero, this yields

$$\begin{aligned}(-i\tau/2) \sin \Omega\tau \Delta_{T_1} \nabla_{T_1} V_1 + \cos \Omega\tau \delta_{T_2} V_1 \\ + (ih^2/2) \left[\frac{\eta}{h} \sin kh + \frac{2\gamma}{h^3} (2 \sin 2kh - \sin kh)\right] \\ \times \Delta_{X_1} \nabla_{X_1} V_1 + \zeta \left[\frac{i \sin kh}{h} V_0 V_1 \right. \\ + \frac{i}{3h} \{c_1(1 + 2 \cos kh) \sin 2kh \\ \left. - c_2(1 + 2 \cos 2kh) \sin kh\} V_1^* V_2 \right] = 0. \qquad (134)\end{aligned}$$

When $l < 4$, the expansion (86) contains only the terms for which $|r| \leqslant 1$, so that Eqs. (123) through (134) are sufficient to ensure that (86) satisfies (53) up to terms of $O(\varepsilon^3)$. This corresponds to the cases $m = N/2$ (or $kh = \pi$) and $m = N/3$ (or $kh = 2\pi/3$).

However, for $l \geqslant 4$ we have to proceed with our analysis and consider higher values of r. Putting $r = 2$ and the coefficient of the ε^2-term equal to zero in (116), we obtain, after some manipulation and with the aid of the linear dispersion relation (124),

$$V_2 = \zeta A(h, k, \tau)\, V_1^2, \qquad (135)$$

where

$$A(h, k, \tau) = \frac{-h^2(1 + 2 \cos kh)}{6g(h, k, \tau)} \qquad (136)$$

and

$$\begin{aligned}g(h, k, \tau) = (\cos kh - 1)[\eta h^2 + 2\gamma(2 \cos^2 kh \\ + 2 \cos kh - 1)] - (\cos \Omega\tau - 1) \\ \times [\eta h^2 + 2\gamma(\cos kh - 1)], \qquad (137)\end{aligned}$$

provided that

$$g(h, k, \tau) \neq 0. \qquad (138)$$

Note that since $[l/2] \geqslant 2$ in these cases, $c_1 = 1$.

With the exception of the cases $\sin kh = \pi$ and $\cos kh = -\frac{1}{2}$, which are excluded anyway for $|r| > 1$, we find that if the condition (138) is violated, it is impossible to remove all the $O(\varepsilon^2)$ terms in (116) (except in the trivial case $V_1 \equiv 0$, or the linear case $\zeta = 0$), so that the expansion (86) is only valid up to the $O(\varepsilon)$ terms. Since the condition (138) does not have a continuous counterpart when $k \neq 0$ and $\gamma \neq 0$, its violation should correspond to spurious behavior of the numerical scheme (53). We shall return to this in Section 5.

We may note in passing that $g = 0$ is also the condition for $\exp(2i\theta)$ to satisfy the linear difference equation

$$(\delta_{T_0} + \eta \delta_{X_0} + \gamma \delta_{X_0} \Delta_{X_0} \nabla_{X_0})\, u_j^n = 0. \qquad (139)$$

If we used the discrete counterparts of (9) through (11), this would correspond to irremovable secular terms due to a second resonance phenomenon. This can only be avoided by using a different type of expansion than (86). We will not pursue the matter further in this paper.

Returning to (116) with $r = 2$, we next put the terms in $O(\varepsilon^3)$ equal to zero, which yields the condition

$$\begin{aligned}\left(\frac{2i \sin kh}{h^3}\right) g(h, k, \tau)\, W_{22} \\ = -\cos(2\Omega\tau)\, \delta_{T_1} V_2 - \left[\eta \cos(2kh) \right. \\ \left. + \frac{2\gamma}{h^2} (\cos(4kh) - \cos(2kh))\right] \delta_{X_1} V_2 \\ - (\zeta c_1/3)(\cos kh + 2 \cos 2kh)\, V_1 \delta_{X_1} V_1. \qquad (140)\end{aligned}$$

Obviously this condition cannot be satisfied either if (138) is violated.

We can simplify (140) by using Lemma 3.2. By (119) we may write

$$V_1 \delta_{X_1} V_1 = \tfrac{1}{2} \delta_{X_1}(V_1)^2 + \text{term in } \varepsilon \qquad (141)$$

and, since (140) already corresponds to a term in ε^3, the term in ε may be neglected. Furthermore, from (135) we obtain

$$\begin{aligned}\delta_{T_1} V_2 &= \zeta A h^2 \delta_{T_1}(V_1)^2 \\ &= 2\zeta A h^2 V_1 \delta_{T_1} V_1 \\ &= -2\zeta A h^2 V_g V_1 \delta_{X_1} V_1 \\ &= -\zeta V_g A h^2 \delta_{X_1}(V_1^2), \qquad (142)\end{aligned}$$

where we have used (132) and the fact that Lemma 3.2 is equally valid with respect to divided differences in T_1 and where any terms in ε were again neglected. Using these relations and (135) in (140), we finally find the relation

$$W_{22} = i\zeta\Gamma(h, k, \tau)\, \delta_{X_1}(V_1)^2, \qquad (143)$$

where

$$2 \sin kh\, g(h, k, \tau)\, \Gamma(h, k, \tau)/h^3 = \Lambda \left[-V_g \cos(2\Omega\tau) + \eta \cos 2kh + \frac{2\gamma}{h^2} (\cos 4kh - \cos 2kh) \right] + (\cos kh + 2 \cos 2kh)/6. \quad (144)$$

For the cases where $l \leqslant 5$, the expansion (86) is truncated at $|r| = 2$, and no further conditions are necessary to ensure satisfaction of (53) up to terms in $O(\varepsilon^3)$. Besides the previously excepted cases $m = N/2$ and $m = N/3$, this now also involves $m = N/4$ $(kh = \pi/2)$, $N/5$ $(kh = 2\pi/5)$, and $2N/5$ $(kh = 4\pi/5)$.

For all other cases, we need also consider (116) with $r = 3$, in which the lowest order terms in ε are $O(\varepsilon^3)$. (Note that in these cases we have $[l/2] > 2$ and therefore $c_2 = 1$.) Putting the coefficient of ε^3 equal to zero, and using (135) and (124), we obtain the relation

$$V_3 = \zeta^2 K(h, k, \tau)\, V_1^3, \quad (145)$$

where

$$K(h, k, \tau) = \frac{-h^2 \Lambda(h, k, \tau)(4 \cos kh - 1)(2 \cos kh + 1)}{3 f(h, k, \tau)} \quad (146)$$

and where

$$f(h, k, \tau) = 4(\cos^2 kh - 1)[\eta h^2 + 2\gamma(4 \cos^3 kh - 1)] - 4(\cos^2 \Omega\tau - 1)[\eta h^2 + 2\gamma(\cos kh - 1)], \quad (147)$$

and where both the conditions (138) and

$$f(h, k, \tau) \neq 0 \quad (148)$$

must be satisfied.

When (138) is satisfied, but not (148), the expansion (86) can satisfy (53) only up to terms in ε^2. This does not have a counterpart in the continuous case either and could be another source of spurious behavior. We will pursue this further in Section 5. Note also that violation of (148) corresponds to $\exp(3i\theta)$ being a solution of the linear difference equation (139). Thus it can also be viewed as a type of third-resonance phenomenon.

We have now managed to express V_2, W_{22}, and V_3 in terms of V_1, and it now remains to do the same for V_0. This is done by finding a solution of (123) which is consistent with the relation (24) in the continuous case (recalling that $B \equiv V_0$).

Similar to the continuous case, V_0 is not uniquely determined by (123), and we are free to impose the additional and physically acceptable condition that V_0 must also satisfy (132), i.e.,

$$\delta_{T_1} V_0 = -V_g \delta_{X_1} V_0. \quad (149)$$

Substituting into (123) we find that

$$\delta_{X_1} \left[V_0 - \frac{\zeta c_1(\cos kh + 2)\, |V_1|^2}{3(V_g - \eta)} \right] = 0, \quad (150)$$

provided that

$$\eta \neq V_g. \quad (151)$$

This then leads to the relation

$$V_0 = \frac{\zeta c_1(\cos kh + 2)\, |V_1|^2}{3(V_g - \eta)}, \quad (152)$$

which is seen to be consistent with (24) in the limit where $kh \to 0$, $h \to 0$, and $\tau \to 0$.

Note that the condition (151), like (138) and (148), does not have a discrete counterpart either, and its violation could therefore also lead to spurious behavior of the solution of (53). To get a better idea of what it means if (151) is violated, note that if $\eta = V_g$, then both V_0 and V_1 satisfy the same difference equation,

$$\delta_{T_1} u + \eta \delta_{X_1} u = 0, \quad (153)$$

with the result that we are no longer able to find a specific relation between V_0 and V_1; we can only conclude from (123) that

$$\delta_{X_1}(|V_1|^2) = 0 \quad (154)$$

and that V_0 and V_1 both represent a motion at velocity η along the discrete grid.

In Section 5 we show that violation of (151) does indeed lead to spurious behavior.

To conclude our analysis, we substitute (135) and (152) into (134) to obtain the following difference equation for V_1:

$$(-i\tau/2) \sin \Omega\tau \Delta_{T_1} \nabla_{T_1} V_1 + \cos \Omega\tau \delta_{T_2} V_1 + (ih^2/2) \left[\frac{\eta}{h} \sin kh + \frac{2\gamma}{h^3} (2 \sin 2kh - \sin kh) \right] \times \Delta_{X_1} \nabla_{X_1} V_1 + \frac{i\zeta^2 \sin kh}{3h} \left[\frac{c_1(\cos kh + 2)}{V_g - \eta} + \Lambda\{2c_1(1 + 2 \cos kh) \cos kh - c_2(1 + 2 \cos 2kh)\} \right] V_1 |V_1|^2 = 0. \quad (155)$$

In all but the few cases noted above, $l > 5$, so that $c_1 = c_2 = 1$. We may then rewrite (155) in the form

$$\begin{aligned}(-i\tau/2)\sin\Omega\tau\Delta_{T_1}\nabla_{T_1}V_1 + \cos\Omega\tau\delta_{T_2}V_1 \\ + (ih^2/2)\left[\frac{\eta}{h}\sin kh + \frac{2\gamma}{h^3}(2\sin 2kh - \sin kh)\right] \\ \times \Delta_{X_1}\nabla_{X_1}V_1 + \frac{i\zeta^2\sin kh}{3h}\left[\frac{\cos kh + 2}{V_g - \eta}\right. \\ \left. + \Lambda(1 + 2\cos kh)\right] V_1 |V_1|^2 = 0. \qquad (156)\end{aligned}$$

It is easy to see that in the limit when h, τ, kh, $\Omega\tau \to 0$,

$$\Lambda \to \frac{1}{6\gamma k^2} \qquad (157)$$

and

$$V_g \to \eta - 3\gamma k^2, \qquad (158)$$

the vanishing terms being second order in h and τ. Thus, in this limit (156) becomes the nonlinear Schrödinger equation

$$(V_1)_{T_2} + 3i\gamma k(V_1)_{X_1X_1} - \frac{i\zeta^2}{6\gamma k} V_1 |V_1|^2 = 0, \qquad (159)$$

which is exactly Eq. (25) with V_1 instead of A. Thus (156) is a second-order finite difference scheme for the nonlinear Schrödinger equation (159). However, together with (132) it also describes the behavior of the variable amplitude V_1 in (86).

Note also that

$$\Gamma(h, k, \tau) \to \frac{1}{6\gamma k^3} \qquad (160)$$

and

$$K(h, k, \tau) \to \frac{1}{48\gamma^2 k^4} \qquad (161)$$

in the limit when h, kh, τ, $\Omega\tau \to 0$, so that the expressions for V_0, V_2, V_3, and W_{22} are also consistent with those given by (24), (38), (40), and (39), respectively, for the continuous case.

We have therefore shown that, if $l > 5$, then the expansion (86) becomes, up to the $O(\varepsilon^3)$ terms,

$$\begin{aligned}u_j^n = \varepsilon(V_1 e^{i\theta} + V_1^* e^{-i\theta}) + \zeta\varepsilon^2\left\{\Lambda[V_1^2 e^{2i\theta} + (V_1^*)^2 e^{-2i\theta}]\right. \\ \left. + \frac{(\cos kh + 2)|V_1|^2}{3(V_g - \eta)}\right\} \\ + \zeta\varepsilon^3\{c_3\zeta K[V_1^3 e^{3i\theta} + (V_1^*)^3 e^{-3i\theta}] \\ + i\Gamma[\delta_{X_1}(V_1)^2 e^{2i\theta} - \delta_{X_1}(V_1^*)^2 e^{-2i\theta}]\}, \qquad (162)\end{aligned}$$

provided that the conditions (138), (148), and (151) are satisfied and where V_1 satisfies (156) and (132).

4. THE MODULATION EQUATIONS FOR SAW-TOOTHED WAVES

The modulation equations derived in the previous section, as well as the expansion (86) take on their simplest forms in the case where $m = N/2$ and $kh = \pi$. This corresponds to a carrier wave with wavelength $2h$, which is the smallest wavelength that can be resolved on the grid we are using. In this case $l = 2$, so that (86) becomes

$$u_j^n = \varepsilon^2 V_0 + \frac{\varepsilon}{2}[V_1^* e^{-i\theta} + V_1 e^{i\theta}], \qquad (163)$$

where

$$\theta = \pi j - \Omega\tau n \qquad (164)$$

and, according to (124),

$$\Omega\tau = 0 \quad \text{or} \quad \Omega\tau = \pi. \qquad (165)$$

Thus the two exponentials in (163) are real and both equal to either $(-1)^j$ or $(-1)^{j+n}$. We may therefore regard the function V_1 as real as well.

From (133) we find that the linear group velocity is given by

$$V_g = \begin{cases} +(4\gamma/h^2 - \eta) & \text{if } \Omega\tau = 0 \\ -(4\gamma/h^2 - \eta) & \text{if } \Omega\tau = \pi. \end{cases} \qquad (166)$$

From (152) we obtain V_0,

$$\begin{aligned}V_0 &= \frac{\zeta V_1^2}{6(V_g - \eta)} \\ &= \begin{cases} \zeta V_1^2 h^2/[12(2\gamma - \eta h^2)] & \text{if } \Omega\tau = 0 \\ -\zeta V_1^2 h^2/(24\gamma) & \text{if } \Omega\tau = \pi, \end{cases} \qquad (167)\end{aligned}$$

provided that, in the case where $\Omega\tau = 0$, $\eta \neq 2\gamma/h^2$ (which is the condition $V_g \neq \eta$ mentioned in the previous section). In the next section we shall show what happens when $\eta \approx 2\gamma/h^2$.

The only remaining equations from the previous section, which is relevant here, are (132) and (134). The latter takes the very simple form

$$\delta_{T_2} V_1 = 0, \qquad (168)$$

and into the former we have to substitute one of the two possible expressions for V_g given by (166). The difference equations for V_1 are therefore *linear*, so that modulations of a saw-toothed carrier wave can be regarded as a linear phenomenon.

Taking all of the above into account, we conclude that (163) can take on one of two forms,

$$u_j^n = \frac{\zeta \varepsilon^2 V_1^2 h^2}{12(2\gamma - \eta h^2)} + \varepsilon V_1 (-1)^j, \tag{169}$$

where

$$\delta_{T_1} V_1 + (4\gamma/h^2 - \eta)\, \delta_{X_1} V_1 = 0, \tag{170}$$

provided that $\eta \neq 2\gamma/h^2$, or

$$u_j^n = \frac{-\zeta V_1^2 \varepsilon^2 h^2}{24\gamma} + \varepsilon V_1 (-1)^{j-n}, \tag{171}$$

where

$$\delta_{T_1} V_1 - (4\gamma/h^2 - \eta)\, \delta_{X_1} V_1 = 0. \tag{172}$$

Since both (132) and (168) are linear, we can expand V_1 as

$$V_1 = \sum_{r=0}^{N/2} \alpha_r \cos \phi_r, \tag{173}$$

where

$$\phi_r = K_r \varepsilon h j - w_1 \varepsilon n \tau - w_2 \varepsilon^2 n \tau + \beta_r \tag{174}$$

and where we must require that

$$K_r \varepsilon h = 2\pi r/N, \qquad r = 0, 1, ..., N/2, \tag{175}$$

to ensure a solution of (53) which satisfies the periodic conditions. If we put this expansion into (132) and (168), we find the following two dispersion relations:

$$\sin w_1 \varepsilon \tau = (\tau/h)\, V_g \sin K \varepsilon h, \tag{176}$$

$$\sin w_2 \varepsilon^2 \tau = 0. \tag{177}$$

Equation (177) implies that $w_2 \varepsilon^2 \tau$ can be either 0 or π, so that (173) must be modified as

$$V_1 = \sum_{r=0}^{N/2} [\alpha_{r1} + (-1)^n \alpha_{r2}] \cos(K_r \varepsilon h j - w_1 \varepsilon n \tau + \beta_r). \tag{178}$$

As in the case of (124), a standard von Neumann stability analysis performed on (132) will yield the stability condition

$$(\tau/h)\, |(4\gamma/h^2 - \eta) \sin K_r \varepsilon h| = |\sin w_1 \varepsilon \tau| \leqslant 1, \qquad r = 0, ..., N/2, \tag{179}$$

from which we obtain the following necessary and sufficient condition for the stability of (132) in the case of a saw-toothed carrier wave:

$$\tau \leqslant \frac{h^3}{4\gamma - \eta h^2}. \tag{180}$$

Depending on the relative values of γ and η, this condition can be slightly more severe than (130) (e.g., when $\eta = 0$).

Finally, we note that (169) and (171) are two particular solutions (up to $O(\varepsilon^3)$) of (53) with saw-toothed carrier waves. We can combine these two cases into a single solution by replacing the expansion (163) with

$$u_j^n = \varepsilon^2 (V_0^{(1)} + (-1)^n V_0^{(2)}) + \varepsilon [V_1^{(1)} (-1)^j + V_1^{(2)} (-1)^{j-n}], \tag{181}$$

where $V_0^{(i)}$ and $V_1^{(i)}$ are real functions of X_1, T_1, and T_2.

Using the techniques of the previous section, we substitute (181) into (53), noting that

$$\delta_{X_0}(-1)^j = \delta_{X_0}(-1)^{j-n} = \delta_{T_0}(-1)^j = \delta_{T_0}(-1)^{j-n} = 0,$$

$$\Delta_{X_0}(-1)^j = -(2/h)(-1)^j = \Delta_{X_0}(-1)^{j-n},$$

and

$$\nabla_{X_0}(-1)^j = +(2/h)(-1)^j = \nabla_{X_0}(-1)^{j-n},$$

and obtain a system of equations by first putting the coefficients of each of $(-1)^j$, $(-1)^{j-n}$, $(-1)^n$, and $(-1)^0$ equal to zero, and then also the coefficients of ε, ε^2, and ε^3 in each of the resulting equations. This yields the following set of equations:

$$\delta_{T_1} V_1^{(1)} + (4\gamma/h^2 - \eta)\, \delta_{X_1} V_1^{(1)} = 0, \tag{182}$$

$$\delta_{T_1} V_1^{(2)} - (4\gamma/h^2 - \eta)\, \delta_{X_1} V_1^{(2)} = 0, \tag{183}$$

$$\delta_{T_2} V_1^{(1)} = \delta_{T_2} V_1^{(2)} = 0, \tag{184}$$

$$(\delta_{T_1} + \eta \delta_{X_1})\, V_0^{(1)} = -\frac{\zeta}{6} \delta_{X_1} [(V_1^{(1)})^2 + (V_1^{(2)})^2], \tag{185}$$

$$(\delta_{T_1} - \eta \delta_{X_1})\, V_0^{(2)} = \frac{\zeta}{3} \delta_{X_1} [V_1^{(1)} V_1^{(2)}]. \tag{186}$$

If we make the physically reasonable assumption that $V_0^{(i)}$ satisfies the same equations (182) and (183) as $V_1^{(i)}$, for $i = 1, 2$, we can, similarly as in the previous section, express $V_0^{(1)}$ and $V_0^{(2)}$ in terms of $V_1^{(1)}$ and $V_1^{(2)}$. Then (181) becomes

$$u_j^n = \frac{\zeta h^2}{12} \left[\frac{(V_1^{(1)})^2 + (V_1^{(2)})^2}{2\gamma - \eta h^2} + (-1)^n \frac{V_1^{(1)} V_2^{(2)}}{\gamma} \right] + V_1^{(1)} (-1)^j + V_1^{(2)} (-1)^{j-n}, \tag{187}$$

provided that $\eta \neq 2\gamma/h^2$.

Note that for this more general type of solution, the stability condition (180) is still valid, as well as the condition $\eta \neq 2\gamma/h^2$.

5. SOME NUMERICAL EXPERIMENTS

In this section we report the results of several numerical experiments. We are especially interested in the cases where the conditions (138), (148), and (151) are violated, since in these cases our multiple scales analysis breaks down for all but a few of the higher carrier wave modes.

It should be recalled that, since ε is supposed to be small in (86) and (162), we are really considering solutions of (53) which are approximately of the form

$$u_j^n = \varepsilon V_1 e^{i\theta} + \varepsilon V_1^* e^{-i\theta}, \tag{188}$$

with θ given by (84) and (124) and where V_1 must satisfy (132) and (155).

In the case of carrier wave numbers for which one or more of the conditions (138), (148), or (151) are violated, the higher order terms in ε may become comparable to, or even dominate, the first-order terms to such an extent that (188) can no longer be considered to be true. This should show up in numerical experiments.

Let us first consider condition (138): First of all, we note that, if we consider carrier wave numbers k which are not near to zero, violation of (138) is equivalent to the equation

$$\eta h^2 + 4\gamma \cos kh(\cos kh + 1) - 2\gamma = O(\tau^2). \tag{189}$$

Since, from (130), linear stability requires that $\tau = O(h^3)$, we can obtain a very good approximation if we neglect the $O(\tau^2)$ term in (189), so that violation of condition (138) will occur for carrier wave numbers approximately satisfying

$$\cos kh = \tfrac{1}{2}[-1 \pm (3 - \eta h^2/\gamma)^{1/2}]. \tag{190}$$

If we consider the semi-discretized scheme only, by letting $\tau \to 0$, Eq. (190) becomes exactly equivalent to

$$g(h, k, \tau) = 0. \tag{191}$$

Thus both the scheme (53) and its semidiscretized version should show some spurious behavior at and near carrier wave numbers satisfying (190). To see what such behavior may be, we performed numerical experiments in which we assigned the following values to the parameters in (53):

$$\begin{aligned} L &= 2.0 \\ N &= 100 \qquad \text{(so that } h = 0.02) \\ \tau &= 0.001 \\ \gamma &= 0.0005 \\ \zeta &= 1.0. \end{aligned} \tag{192}$$

TABLE I

Mode Numbers (m) and Values of η for Which Eq. (191) Is Satisfied

m	η	m	η	m	η
1	−9.891356	18	−0.549300	35	3.711460
2	−9.602409	19	−0.025281	36	3.655535
3	−9.191731	20	0.474235	37	3.579612
4	−8.707842	21	0.946053	38	3.487670
5	−8.178529	22	1.387138	39	3.383788
6	−7.620031	23	1.794708	40	3.272076
7	−7.041867	24	2.166328	41	3.156593
8	−6.450363	25	2.500000	42	3.041272
9	−5.849971	26	2.794243	43	2.929843
10	−5.244235	27	3.048139	44	2.825760
11	−4.636170	28	3.261374	45	2.732136
12	−4.028562	29	3.434246	46	2.651690
13	−3.424130	30	3.567650	47	2.586701
14	−2.825615	31	3.663056	48	2.538972
15	−2.235856	32	3.722460	49	2.509809
16	−1.657811	33	3.748336	50	Not applicable
17	−1.094562	34	3.743582		

We wished to assign to η a value which would cause (191) to be satisfied for one particular value of the wave number k. Table I shows such values of η for each of the values of $k = m\pi$, $m = 1, \ldots, 50$. We assigned to η the value of

$$\eta = 2.794243, \tag{193}$$

which causes the 26th carrier mode (i.e., $k = 26\pi$) to satisfy (191).

We used the initial condition

$$u_j^0 = \varepsilon \cos(h\pi j) \cos(m\pi hj) \tag{194}$$

and ran the scheme (53) for 100,000 time steps in double precision, after each 2000 time steps taking a discrete Fourier transform of the u_j^n. The time evolution of each Fourier mode was then plotted as a three-dimensional graph. Figures 1 through 5 show the results when $\varepsilon = 0.01$

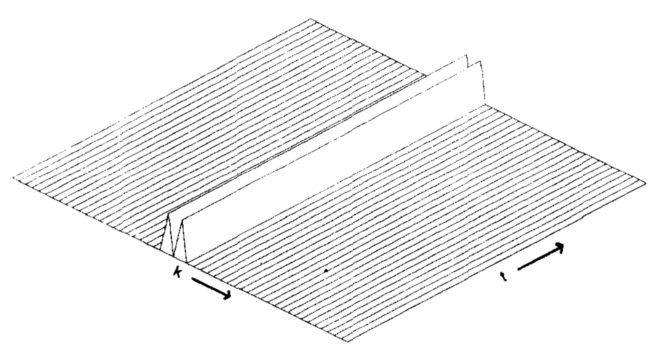
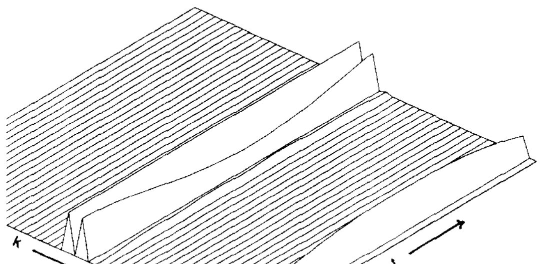


FIG. 1. Time evolution of Fourier modes of the solution of (53). Initial data is (194), parameter values given by (192), $\varepsilon = 0.01$, $\eta = 2.794243$, and $m = 24$.



[redacted]

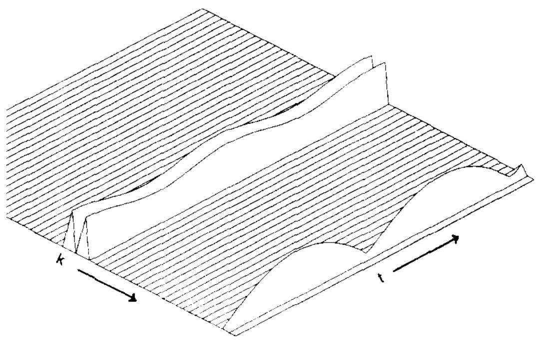


FIG. 3. Same as in Fig. 1, but with $m=26$.

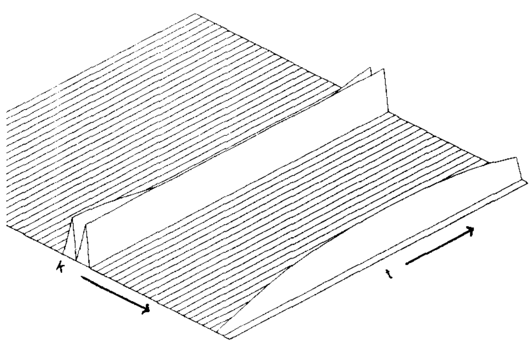


FIG. 4. Same as in Fig. 1, but with $m=27$.

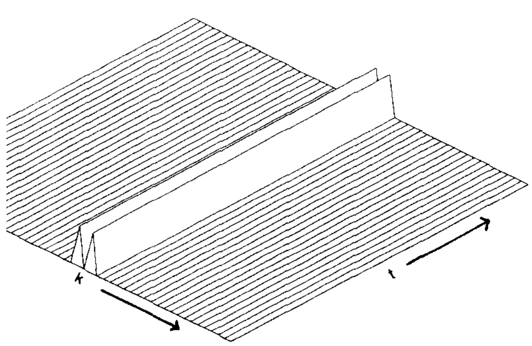


FIG. 5. Same as in Fig. 1, but with $m=28$.

and $m=24$, 25, 26, 27, and 28, respectively. Since (194) can be written as

$$u_j^0=(\varepsilon/2)\cos[\pi(m+1)\,hj] + (\varepsilon/2)\cos[\pi(m-1)\,hj], \qquad (195)$$

there are initially only the two modes with wave numbers $(m-1)\pi$ and $(m+1)\pi$ present. In the cases of $m=24$ and $m=28$ these remain the only two modes throughout the calculation. Any nonlinear effects here are so small that they [redacted] a small nonlinear perturbation which remains small.

When $m=25$, 26, and 27, however, we see significant nonlinear activity. In each case the 48th mode is excited and seems to exhibit a near periodic behavior in time, corresponding to a similar type of behavior of one or both of the two initial modes. In total the solution shows almost recurrence of the initial state, with a much shorter recurrence time in the case $m=26$ than in the other two cases. It is interesting to note that the 2θ term in (86) contains a carrier wave which, on a much finer grid, would have had a wave number of 52π. On our grid, however, this would alias to a mode with wave number 48π, corresponding exactly to the excited mode. This makes sense, since $g\approx 0$ would tend to magnify the amplitude of this particular carrier wave, and in this case such magnification is sufficient to show up clearly on our graphs.

In other experiments, with larger values of ε, even more pronounced nonlinear activity was noted in the vicinity of $m=26$. Thus at and near $m=26$ the solution of (53) deviates sharply from a modulated harmonic wave described by (188). This is only to be expected, however, since our multiple scales analysis, which is based on the assumption that we have a solution approximated by (188) for small ε, breaks down at $m=26$.

We next turn to an investigation of the condition (148). From (147) we see that, for k not near zero, violation of this condition is equivalent to

$$\eta h^2+2\gamma(4\cos^3 kh-1)=O(\tau^2), \qquad (196)$$

and, as before, we can obtain a very good approximation by neglecting the $O(\tau^2)$ term. Thus the carrier wave number k for which

$$f(h,k,\tau)=0 \qquad (197)$$

is approximately given by

$$\cos kh=\left(1-\frac{\eta h^2}{2\gamma}\right)^{1/3}, \qquad (198)$$

TABLE II

Mode Numbers (m) and Values of η for Which Eq. (197) Is Satisfied

m	η	m	η	m	η
1	−7.441332	18	1.728113	35	4.530748
2	−7.265689	19	2.001123	36	5.089915
3	−6.978321	20	Not applicable	37	5.707824
4	−6.587092	21	2.346194	38	6.373714
5	−6.102629	22	2.434208	39	7.074485
6	−5.537959	23	2.480312	40	Not applicable
7	−4.908064	24	2.497525	41	8.519188
8	−4.229370	25	Not applicable	42	9.229372
9	−3.519186	26	2.502476	43	9.908066
10	−2.795120	27	2.519688	44	10.53796
11	−2.074483	28	2.565793	45	11.10263
12	−1.373713	29	2.653806	46	11.58709
13	−0.707823	30	2.795085	47	11.97832
14	−0.089914	31	2.998867	48	12.26569
15	0.469252	32	3.271888	49	12.44133
16	0.961586	33	3.618087	50	Not applicable
17	1.381915	34	4.038415		

TABLE III

Mode Numbers (m) and Values of η for Which $V_g = \eta$

m	η	m	η	m	η
1	−9.963392	18	−5.007626	35	0.441773
2	−9.857949	19	−4.657241	36	0.690761
3	−9.694504	20	−4.307267	37	0.926692
4	−9.485698	21	−3.958300	38	1.148775
5	−9.242728	22	−3.610948	39	1.356242
6	−8.974270	23	−3.265837	40	1.548361
7	−8.686666	24	−2.923612	41	1.724440
8	−8.384472	25	−2.584943	42	1.883831
9	−8.070973	26	−2.250526	43	2.025942
10	−7.748589	27	−1.921081	44	2.150234
11	−7.419139	28	−1.597355	45	2.256232
12	−7.084041	29	−1.280115	46	2.343528
13	−6.744425	30	−0.970155	47	2.411782
14	−6.401232	31	−0.668282	48	2.460728
15	−6.055269	32	−0.375318	49	2.490172
16	−5.707248	33	−0.092097	50	2.500000
17	−5.357828	34	−0.180546		

this approximation becoming exact for the semidiscretized version of (53).

Using the same values for the various parameters given in (192), we sought to choose η in such a way that (197) is satisfied for some particular value of k. Table II shows the various values of η for which each of $k = m\pi$, $m = 1, ..., 50$, would satisfy (197).

We repeated the previous type of numerical experiment with the initial condition (194), using $\varepsilon = 0.01$, $\varepsilon = 0.1$, and even higher values of ε, and choosing various values for η from Table II. However, even when we took m equal to the value for which (197) is satisfied in each case, our graphs always showed a two-mode solution similar to that in Figs. 1 and 5. Any spurious nonlinear effects were just too small to show up on our graphs.

This is not entirely unexpected, since violation of condition (148) should have an effect on the $O(\varepsilon^3)$ terms in (86), instead of the $O(\varepsilon^2)$ terms when (138) is violated. Even though these terms are magnified at and near the critical wave number, such magnification never becomes big enough to be noticed. Condition (148) does not, therefore, seem to be very important in practice.

We next turn to condition (151). This condition is violated when $V_g = \eta$, or

$$\left(\frac{1}{\cos \Omega\tau}\right)\left[\eta \cos kh + \frac{2\gamma}{h^2}(\cos 2kh - \cos kh)\right] - \eta = 0. \quad (199)$$

For k not near to zero, this is equivalent to

$$\eta + \frac{2\gamma}{h^2}(2\cos kh + 1) = O(\tau^2), \quad (200)$$

and once more we can obtain a very good approximation by neglecting the $O(\tau^2)$ term. Thus (199) is satisfied approximately when the wave number k is given by

$$\cos kh = -\left(1 + \frac{\eta h^2}{2\gamma}\right)\Big/2. \quad (201)$$

Note that when $\eta = 0$, this corresponds to $kh = 2\pi/3$, or $m = N/3$.

Using the parameters given by (192), we generated Table III, which shows the various values of η for which each of $k = m\pi$, $m = 1, ..., 50$ would satisfy $\eta = V_g$. We then assigned to η the value

$$\eta = 1.356242, \quad (202)$$

which would cause η to be equal to V_g for $m = 39$. As before, we used the initial condition (194) with $\varepsilon = 0.01$ and ran scheme (53) for 100,000 time steps in double precision, taking a discrete Fourier transform after every 2000 steps. The results for $m = 38$, 39, and 40 are shown in Figs. 6–8. For $m = 38$ and $m = 40$ no significant deviation from a two-mode solution is seen; i.e., the approximation (188) is very good here. However, when $m = 39$ a mode with wave number 2π is excited, and its amplitude shows a near periodic behavior in time, corresponding to a similar behavior in the amplitudes of both initial modes. In this case, therefore, a solution with a spurious *low* wave number mode is obtained.

In another series of experiments we considered the case $m = 50$, corresponding to $kh = \pi$ or the saw-toothed carrier wave, in which case the equation $V_g = \eta$ becomes $\eta = 2\gamma/h^2$,

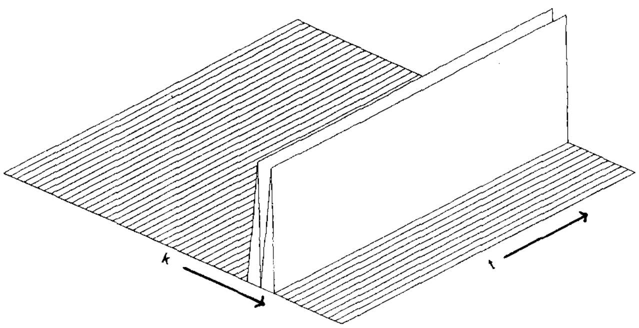


FIG. 6. Time evolution of Fourier modes of the solution of (53). Initial data is (194), parameter values given by (192), $\varepsilon = 0.01$, $\eta = 1.356242$, and $m = 38$.

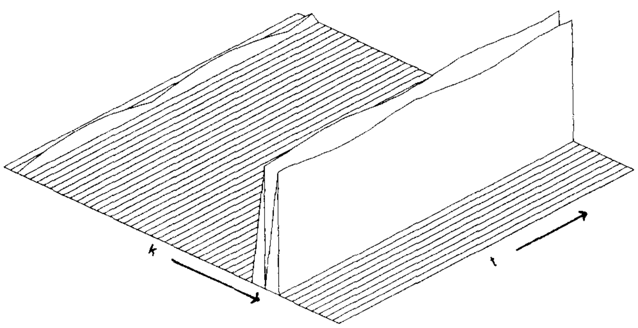


FIG. 7. Same as in Fig. 6, but with $m = 39$.

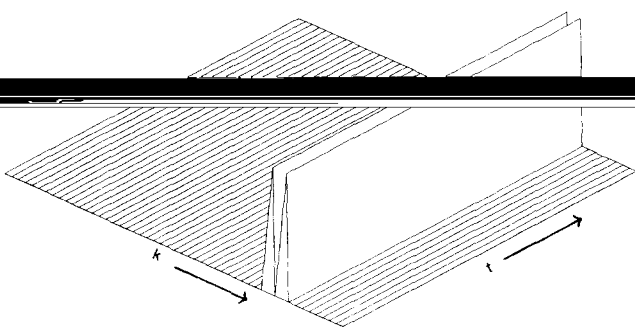


FIG. 8. Same as in Fig. 6, but with $m = 40$.

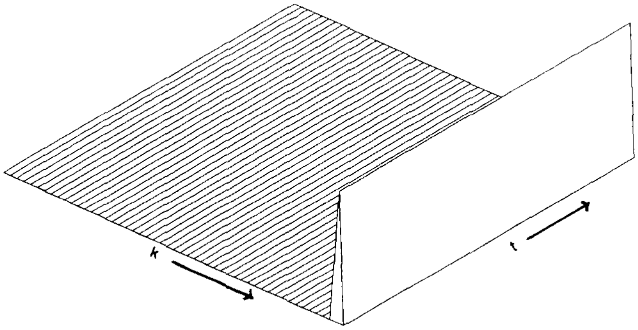


FIG. 9. Time evolution of Fourier modes of the solution of (53). Initial data is (194), parameter values given by (192), $\varepsilon = 0.01$, $m = 50$, and $\eta = 2.4$.

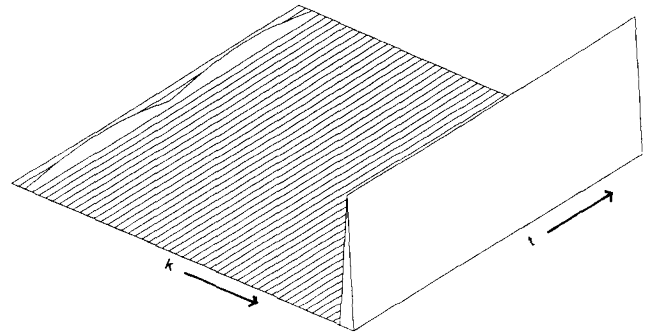


FIG. 10. Same as in Fig. 9, but with $\eta = 2.5$.

as explained in the previous section. For the choice of parameters given by (192), this critical value of η is given by $\eta = 2.5$. We used the initial condition (194) once more, which is now of the form

$$u_j^0 = 0.01(-1)^j \cos(h\pi j). \tag{203}$$

Referring to (195), we see that on our grid this would show up as a single mode solution when performing a discrete Fourier transform, since the mode with wave number 51π aliases onto the mode with wave number 49π.

We ran the scheme (53) with this initial data, again for 100,000 time steps as before, performing discrete Fourier transforms every 2000 steps and using various values of η. The results for $\eta = 2.4$, 2.5, and 2.6 are shown in Figs. 9, 10, and 11, respectively. Figures 9 and 11 show no significant extra modes being created. Approximation (188) is essen- [illegible] that a mode with wave number 2π and a significant amplitude is excited, thus indicating a solution which deviates from (188).

The numerical results in this section therefore show that the multiple scales analysis described in this paper correctly indicate the carrier wave numbers for which the behavior of the solution of scheme (53) deviates sharply from a small modulated harmonic wave. It also shows that condition (148) is much less important than conditions (138) and (151).

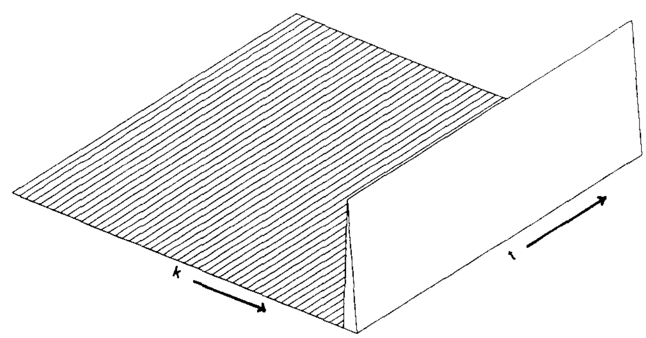


FIG. 11. Same as in Fig. 9, but with $\eta = 2.6$.

The multiple scales analysis for the continuous equation (1) does not indicate such behavior, so that we have identified some cases where the numerical solution does not resemble the solution of (1).

6. CONCLUSIONS

The discrete multiple scales technique described in this paper is a powerful tool for the examination of the modulation properties of equations such as the KdV equation (1).

In this paper we showed that this technique can identify modes of the carrier wave of the envelope of a small, modulated harmonic wave, for which the solution of the numerical solution deviates sharply from that of the KdV equation.

We also obtained a discrete version of the nonlinear Schrödinger equation, which describes the modulation properties of solutions of (53) which can be expanded in the form (86). This is exactly analogous to the continuous case.

Obviously these methods of analysis can be applied to other discretizations of the KdV equation, as well as to discretizations of other dispersive wave equations, such as the modified KdV (MKdV) equation. This type of analysis would also be useful when dealing with models which are discrete to start with.

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